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Rigid Motions in Relativity

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Starting with a simple characterization of pairs of rigidly joined straight world lines, successive generalizations are obtained up to the most general case which allows us to establish the various definitions of rigid motions in relativity (both special and general).

1. INTRODUCTION

Since Born (1909) gave the first definition, many other definitions of rigidity have been given, some of them in order to overcome objections as to the possibility of introducing the rigid body concept, and others so as to alleviate the restrictive conditions imposed by Born's definition.

The problem was conceptually clarified by the concept of kinematical rigidity (Synge, 1972), but the problem of deriving the various admissible definitions still remains. This paper is a new approach to this question.

2. PRELIMINARY DEFINITIONS

Let us consider a four-dimensional flat hyperbolic manifold of class $\mathcal{C}^2$ with time orientation. As is usual, we will call the points of this manifold events and the timelike curves, world lines.

In an orthonormal and global coordinate system, $(0, \mathbf{e}_i)$, whose existence is ensured by the structure considered, let us choose for the metric the expression

$$ds^2 = -\sum_{\alpha=1}^{3} (dx^\alpha)^2 + (dx^4)^2 \tag{1}$$

and also the vector $\mathbf{e}_4$ to be oriented towards the future.

Definition 1. An *I*-like reference system (IRS) is a congruence of straight parallel world lines.

We denote by I_0 the IRS in which the direction of the world lines is indicated by $\mathbf{e}_4$. Next, we give the following definition.

Definition 2. Two arbitrary world lines, L_1, L_2, of an IRS, I_R, form the end points of a measuring rod at rest with respect to I_R (MRI_R).

It is clear from Definition 2 that if $x'=x^r(s)$ and $x''=x''(s')$ are two natural representations of two world lines L_1 and L_2 of an IRS, I_R, and if x^r and x'' are the respective coordinates of two events 0_1 and 0_2 of L_1 and L_2 in such a way that the vector $\eta^r=x''^r-x^r$ is orthogonal to L_1 in 0_1, then η^r is orthogonal to L_2 in 0_2. On the other hand, it can be verified that η^r has a constant modulus (which is positive, because it is spacelike).

The measuring rod, which we denote by (L_1, L_2), is formed by all the world lines L: $x''^r=x''^r(s)$ of I_R which satisfy

$$x''^r=x^r(s)+\lambda\eta^r \qquad (0\leqslant\lambda\leqslant 1) \tag{2}$$

For $\lambda=0$, $\lambda=1$, the world lines obtained are the end points L_1 and L_2 of the MRI_R under consideration.

Definition 3. The length of an MRI_R, (L_1, L_2), in I_0 is the modulus of the vector η^r.

Definition 4. An MRI_R, (L_1, L_2), with length $(\eta^r\eta_r)^{1/2}$ in I_0, has the $\mathbf{e}_1$ direction if and only if $\eta^\alpha=0$, $(\alpha=2,3)$. (Similarly for $\mathbf{e}_2$ and $\mathbf{e}_3$.)

3. CONSTANT RIGID MOTIONS

Unless we state otherwise we will exclusively use IRSs, $[I_R,(0,\mathbf{e}_i)]$, $[I'_R,(0',\mathbf{e}'_i)]\cdots$ related by restricted Lorentz transformations of the following type:

$$x'^1=(x^1-vt)\gamma, \qquad t'=\left(t-\frac{vx^1}{c^2}\right)\gamma, \qquad x'^2=x^2, \qquad x'^3=x^3$$

$$\left[\gamma=\left(1-\frac{v^2}{c^2}\right)^{-1/2}, x^4=ct\right] \tag{3}$$

(as is known, this is not an essential restriction).

Let $\mathcal{C}_1$ be the class of straight world lines parallel to the 2-flat (x^1, x^4) of a system of the type we are considering. We can now give the following definition:

Definition 5. If $L_1, L_2 \in \mathcal{Q}_1$,

$$(L_1, L_2) \in \mathfrak{R}J_{(1)} \subset \mathcal{Q}_1 \times \mathcal{Q}_1 \Leftrightarrow \exists I_R / L_1, L_2 \in I_R \tag{4}$$

where by means of $\mathfrak{R}J_{(1)}$ we denote the relation "to be rigidly joined."

It can be easily verified that $\mathfrak{R}J_{(1)}$ is a relation of equivalence. On the other hand, if we denote by l_0 the length in I_{0_R} of an MRI_{0_R} in the $\mathbf{e}_1$ direction, it is easy to see the following:

If $L_1 \mathfrak{R}J_{(1)} L_2$, then, in the system $[I_{0_R}, (0, \mathbf{e}_{0_i})]$ in which $L_1, L_2 \in I_{0_R}$, the following conditions

$$\left[\underset{0}{x}{}^1_{(2)}\left(\underset{0}{t}_1\right) - \underset{0}{x}{}^1_{(1)}\left(\underset{0}{t}_2\right)\right] l_0^{-1} = \theta_0$$

$$\underset{0}{x}{}^\beta_{(2)}\left(\underset{0}{t}_1\right) - \underset{0}{x}{}^\beta_{(1)}\left(\underset{0}{t}_2\right) = \underset{0}{\theta}_\beta \qquad \left(\theta_0, \underset{0}{\theta}_\beta \text{ const}, \quad \beta = 2,3\right) \tag{5}$$

hold for arbitrary $\underset{0}{t}_1$ and $\underset{0}{t}_2$.

Theorem 1. If $[I_R, (0, \mathbf{e}_i)]$ is related to $[I_{0_R}, (0, \mathbf{e}_{0_i})]$ by (3) and if we denote by l the length in I_R of the MRI_{0_R} which we have previously determined, then the conditions

$$\left[x^1_{(2)}(t) - x^1_{(1)}(t)\right] l^{-1} = \theta$$

$$x^\beta_{(2)}(t) - x^\beta_{(1)}(t) = \theta_\beta \qquad \left(\theta, \theta_\beta \text{ const}, \quad \beta = 2,3\right) \tag{6}$$

are satisfied in $[I_R, (0, \mathbf{e}_i)]$, where $x^i_{(j)}$ $(i = 1,2,3;\ j = 1,2)$ are the coordinates in $[I_R, (0, \mathbf{e}_i)]$ of the events of two lines L_1 and L_2 which belong to I_{0_R}.

The demonstration is easy because between the coordinates $\underset{0}{x}{}^i_{(j)}$ of the events of L_1 and L_2 in $[I_{0_R}, (0, \mathbf{e}_{0_i})]$ and the coordinates $x^i_{(j)}(t)$ in $[I_R, (0, \mathbf{e}_i)]$ corresponding to the value t, there exists the relation

$$x^1_{(2)} - x^1_{(1)} = \left(\underset{0}{x}{}^1_{(2)} - \underset{0}{x}{}^1_{(1)}\right)\gamma^{-1} \tag{7}$$

Then, using $l = l_0 \gamma^{-1}$, we obtain

$$\theta_0 = \theta, \qquad \underset{0}{\theta}_\beta = \theta_\beta \qquad (\beta = 2,3) \tag{8}$$

i.e., (6) and (5) are of the same type and have the same values for the constants θ and θ_β.

On the other hand, differentiating (6) with respect to t, we have

$$\frac{d}{dt}\left[x^{\alpha}_{(2)}(t)\right]-\frac{d}{dt}\left[x^{\alpha}_{(1)}(t)\right]=0$$

thus, we can state the following theorem:

Theorem 2. If $L_1, L_2 \in \mathcal{C}_1$ and fulfil the conditions (6) in a system $[I_R,(0,\mathbf{e}_i)]$ then $L_1 \mathfrak{R} J_{(1)} L_2$.

Consequently, we can adopt the following definition as equivalent to Definition 5.

Definition 6. If $L_1, L_2 \in \mathcal{C}_1$, $L_1 \mathfrak{R} J_{(1)} L_2$ if, and only if, the conditions (6) are fulfilled.

Then, considering that a world line is the history of a particle, we can give, in kinematical terms, the following definition:

Definition 7. The motion in a system $[I_R,(0,\mathbf{e}_i)]$ of a set of particles of $\mathcal{C}_1$ is a rigid motion when each pair of particles fulfils the conditions (6).

From the Definitions 1, 2, and 7 it follows that the motion in a system $[I_R,(0,\mathbf{e}_i)]$ of any MRI_{0_R} [I_{0_R} related to I_R by (3)] is rigid. Another consequence is that an IRS appears as a class of equivalence in $\mathcal{C}_1$ with respect to $\mathfrak{R} J_{(1)}$.

4. ARBITRARY RECTILINEAR MOTIONS

Let us consider a system $[I_{0_R},(0,\mathbf{e}_{0_i})]$ and let $(\underset{0}{x}{}^{\alpha}_{(1)}, t_0)(\underset{0}{x}{}^{\alpha}_{(2)}, t_0)$ be the coordinates in $[I_{0_R},(0,\mathbf{e}_{0_i})]$ of two arbitrary events of the world lines L_1 and L_2, respectively, in such a way that $L_1, L_2 \in I_{0_R}$. If $[I_R,(0,\mathbf{e}_i)]$ is related to $[I_{0_R},(0,\mathbf{e}_{0_i})]$ by (3) then, in this new system, the events we are considering have the coordinates $(x^{\alpha}_{(1)}, t_1)$ and $(x^{\alpha}_{(2)}, t_2)$ associated to them by

$$\underset{0}{x}{}^{1}_{(\beta)}=\left(x^{1}_{(\beta)}+vt_{\beta}\right)\gamma, \qquad t_0=\left(t_{\beta}+\frac{v}{c^2}x^{1}_{(\beta)}\right)\gamma \qquad (\beta=1,2) \tag{9}$$

From the last two equations in (9) we obtain

$$t_2-t_1=-\frac{v}{c^2}\left[x^{1}_{(2)}(t_2)-x^{1}_{(1)}(t_1)\right] \tag{10}$$

whereas from the first two, we have

$$\underset{0}{x}{}^{1}_{(2)}-\underset{0}{x}{}^{1}_{(1)}=\left[\left(x^{1}_{(2)}-x^{1}_{(1)}\right)+v(t_2-t_1)\right]\gamma \tag{11}$$

or, taking into account (5) and (10),

$$x^1_{(2)}(t_2)-x^1_{(1)}(t_1)=\theta l_0\gamma \tag{12}$$

On the other hand, if the conditions

$$\begin{aligned} x^1_{(2)}(t_2)-x^1_{(1)}(t_1)&=\theta l_0\gamma \\ x^\alpha_{(2)}(t_2)-x^\alpha_{(1)}(t_1)&=\theta_\alpha \\ t_2-t_1&=-(v/c^2)\theta l_0\gamma \qquad (\alpha=2,3) \end{aligned} \tag{13}$$

are verified in $[I_R,(0,\mathbf{e}_i)]$, then the corresponding events are simultaneously observed in $[I_{0_R},(0,\mathbf{e}_{0_i})]$ and we also have $\underset{0}{x}{}^1_{(2)}(t_0)-\underset{0}{x}{}^1_{(1)}(t_0)=\theta l_0$.

Now, consider the class $\mathcal{C}_2(\mathcal{C}_2\supset\mathcal{C}_1)$ of world lines in such a way that their 4-direction in each of their events is parallel to the 2-plane (x^1, x^4) of a given system $[I_R,(0,\mathbf{e}_i)]$. Taking this into account we adopt the following definition:

Definition 8. L_2 is rigidly joined to L_1 (we denote this situation with the notation $L_2\mathfrak{R}J_{(2)}L_1$) if, and only if, each event of L_1, with coordinates $(x^\alpha_{(1)}(t_1), t_1)$, has one (and only one) event of L_2, with coordinates $(x^\alpha_{(2)}(t_2), t_2)$, associated to it by

$$\begin{aligned} x^1_{(2)}(t_2)-x^1_{(1)}(t_1)&=\theta l_0\gamma \\ x^\alpha_{(2)}(t_2)-x^\alpha_{(1)}(t_1)&=\theta_\alpha \\ t_2-t_1&=(u_1)_{t_1}c^{-2}\theta l_0\gamma \qquad \left[\gamma=\left(1-\frac{(u_1)_{t_1}}{c^2}\right)^{-1/2}, \qquad \alpha=2,3\right] \end{aligned} \tag{14}$$

where l_0 is the length of an MRI_R in the $\mathbf{e}_1$ direction and $(u_1)_{t_1}$ is the 3-velocity of L_1 in $t=t_1$.

Theorem 3. If L_1 and L_2 are two lines of $\mathcal{C}_2$ in such a way that $L_2\mathfrak{R}J_{(2)}L_1$ and also that their 4-direction is parallel to $\mathbf{e}_4$ for the value $t=t_0$, then $[(\theta l_0)^2+\theta_2^2+\theta_3^2]^{1/2}$ represents the three-dimensional spatial distance, D_{t_0}, measured in $[I_R,(0,\mathbf{e}_i)]$ for the value t_0, between the particles corresponding to those lines.

This is so because setting $(u_1)_{t_0}=0$ in the last of the equations (14), we have $t_2=t_1=t_0$; and putting this result into the three first equations of (14), we obtain

$$x^1_{(2)}(t_0)-x^1_{(1)}(t_0)=\theta l_0$$
$$x^\alpha_{(2)}(t_0)-x^\alpha_{(1)}(t_0)=\theta_\alpha \qquad (\alpha=2,3) \tag{15}$$

from which we conclude

$$D_{t_0}=\left[(\theta l_0)^2+\theta_2^2+\theta_3^2\right]^{1/2} \tag{16}$$

the relation we wanted to prove.

Theorem 4. $L_2 \mathfrak{R} J_{(2)} L_1$ if and only if $\forall t_1$ the vector $a^r=(x^\alpha_{(2)}(t_2)-x^\alpha_{(1)}(t_1), t_2-t_1)$ satisfies

$$(u_1)_{t_1}\left[x^1_{(2)}(t_2)-x^1_{(1)}(t_1)\right]-c^2(t_2-t_1)=0$$
$$x^\alpha_{(2)}(t_2)-x^\alpha_{(1)}(t_1)=\theta_\alpha \tag{17}$$
$$\left[x^\alpha_{(2)}(t_2)-x^\alpha_{(1)}(t_1), t_2-t_1\right]^2=D^2_{t_0} \qquad (\alpha=2,3)$$

We start by observing that the 4-direction of L_1 in such an event is determined by $((u_1)_{t_1},0,0,1)$. Then, from the last relation (14), we have

$$(u_1)_{t_1}\theta l_0\gamma-c^2(t_2-t_1)=0 \tag{18}$$

and, taking into account the first relation in (14), we have

$$(u_1)_{t_1}\left[x^1_{(2)}(t_2)-x^1_{(1)}(t_1)\right]-c^2(t_2-t_1)=0 \tag{19}$$

On the other hand

$$\left[x^\alpha_{(2)}(t_2)-x^\alpha_{(1)}(t_1), t_2-t_1\right]^2=\frac{(\theta l_0)^2}{\gamma^{-2}}+\theta_2^2+\theta_3^2-\frac{(\theta l_0)^2}{\gamma^{-2}}(u_1)^2_{t_1}=D^2_{t_0} \tag{20}$$

Conversely, if the corresponding events of L_1 and L_2 are linked by the relations (17) then $L_2 \mathfrak{R} J_{(2)} L_1$; because from the last relation in (17) we deduce

$$\left[x^1_{(2)}(t_2)-x^1_{(1)}(t_1)\right]^2-c^2(t_2-t_1)^2=(\theta l_0)^2 \tag{21}$$

and from the first we deduce

$$t_2 - t_1 = \frac{(u_1)_{t_1}}{c^2}\left[x^1_{(2)}(t_2) - x^1_{(1)}(t_1)\right] \tag{22}$$

then, putting this result into (21), we obtain

$$\left[x^1_{(2)}(t_2) - x^1_{(1)}(t_1)\right]^2 - \frac{(u_1)^2_{t_1}}{c^2}\left[x^1_{(2)}(t_2) - x^1_{(1)}(t_1)\right]^2 = (\theta l_0)^2 \tag{23}$$

i.e., the first of the relations (14), which, when substituted in (22) gives us the last relation (14), which is what we wished to demonstrate.

Theorem 5. If $L_1, L_2 \in \mathcal{C}_2$ then $(u_2)_{t_2} = (u_1)_{t_1} \Leftrightarrow (u_1)_{t_1} = k;\ \forall t_1$ (k const)

From the last relation in (14) we have

$$\frac{dt_2}{dt_1} = 1 + \frac{d}{dt_1}\left[\frac{(u_1)_{t_1}}{c^2}\,\frac{\theta l_0}{1-(u_1)^2_{t_1}/c^2}\right] \tag{24}$$

and from the first relation in (14)

$$(u_2)_{t_2}\frac{dt_2}{dt_1} - (u_1)_{t_1} = \frac{d}{dt_1}(\theta l_0 \gamma) \tag{25}$$

By substitution of (24) in (25) we have

$$(u_2)_{t_2} = \left[(u_1)_{t_1} + \frac{d}{dt_1}(\theta l_0 \gamma)\right]\left\{1 + \frac{d}{dt_1}\left[\frac{(u_1)_{t_1}}{c^2}\theta l_0 \gamma\right]\right\}^{-1} \tag{26}$$

and the theorem is demonstrated.

From this theorem it is immediately evident that Definition 8 contains Definition 6 as a particular case. On the other hand, it can be verified without difficulty that $\mathfrak{R}J_{(2)}$ is a relation of equivalence.

Definition 9. The motion of a set of particles of $\mathcal{C}_2$ in a system $[I_R, (0, \mathbf{e}_i)]$ is a rigid motion when each pair of particles fulfils the conditions (14).

5. ARBITRARY MOTIONS

Consider two arbitrary world lines L_1 and L_2 whose equations in a given system $[I_R,(0,\mathbf{e}_i)]$ are, respectively, $\underset{(1)}{x_r}=\underset{(1)}{x_r}(s)$ and $\underset{(2)}{x_r}=\underset{(2)}{x_r}(\bar{s})$, where s and $\bar{s}$ are the separations of those lines. (In this section we will use, for simplicity, Minkowskian coordinates.) Theorem 4 suggests the following generalization.

Definition 10. L_2 is rigidly joined to L_1 (we will use the symbol $L_2\mathfrak{R}J_{(3)}L_1$) when for any event of L_1, with coordinates $\underset{(1)}{x_r}$, there exists a unique event of L_2, with coordinates $\underset{(2)}{x_r}$, in such a way that the vector $\eta_r=\underset{(2)}{x_r}-\underset{(1)}{x_r}$ satisfies

$$\eta_r^2=D^2 \tag{27a}$$

$$\underset{(1)}{\lambda_r}\eta_r=0 \qquad (D\text{ const.}) \tag{27b}$$

where $\underset{(1)}{\lambda_r}=d\underset{(1)}{x_r}/ds$.

The relation $\mathfrak{R}J_{(3)}$ is symmetric because from (27a) we have $(-\eta_r)^2=D^2$ and on the other hand, because of the fact that $\underset{(2)}{x_r}=\underset{(1)}{x_r}+\eta_r$ we have the relation

$$\eta_r\underset{(2)}{\lambda_r}=\eta_r\left(\frac{d\underset{(1)}{x_r}}{ds}+\frac{d\eta_r}{ds}\right)\frac{ds}{d\bar{s}}=0 \tag{28}$$

where we use $\underset{(2)}{\lambda_r}$ to denote the vector $d\underset{(2)}{x_r}(\bar{s})/d\bar{s}$.

Nevertheless, in contrast to the previous situations, $\mathfrak{R}J_{(3)}$ is not transitive, as we can see from the following example.

Let L_1, L_2, and L_3 be three world lines whose equations in a given system $[I_R,(0,\mathbf{e}_i)]$ are

$$L_1\begin{cases}x_1=r\cos\omega t\\ x_2=r\sin\omega t,\\ x_3=0\end{cases}\qquad L_2\{x_\alpha=0,\ \alpha=1,2,3\},\qquad L_3\begin{cases}x_1=r+h\\ x_2=x_3=0\end{cases} \tag{29}$$

where r, ω, and h are positive constants that satisfy the conditions

$$h>r, \qquad 0<r<c/\omega \tag{30}$$

Then $L_2 \mathfrak{R} J_{(3)} L_3$ because for any $\underset{(2)}{x_r} \in L_2$ the vector $\eta_r=(r+h,0,0,0)$ satisfies (27a) and (27b). Also $L_1 \mathfrak{R} J_{(3)} L_2$ because if $\underset{(1)}{x_r}$ is an arbitrary event of L_1 then, considering the event $\underset{(2)}{x_r}=(0,0,0,x_4)$, we have $\left|\underset{(2)}{x_r}-\underset{(1)}{x_r}\right|^2=r^2$; whereas, knowing that $\underset{(1)}{\lambda_r}=(-r\omega\sin\omega t,\ r\omega\cos\omega t,\ 0,\ ic)$, we have $\underset{(1)}{\lambda_r}(\underset{(2)}{x_r}-\underset{(1)}{x_r})=r^2\omega(\sin\omega t\cos\omega t-\sin\omega t\cos\omega t)=0$. Nevertheless L_1 is not rigidly joined to L_3. In order to verify this, let us consider the event $(\underset{(1)}{x_r})_1=(r,0,0,0)$ of L_1; then $\underset{(1)}{\lambda_r}$ takes the value $(0,r\omega,0,ic)$ in $(\underset{(1)}{x_r})_1$. It is then easy to find a unique event $(\underset{(3)}{x_r})_1$ of L_3 in such a way that $(\underset{(3)}{x_r})_1-(\underset{(1)}{x_r})_1$ is orthogonal to L_1 in $(\underset{(1)}{x_r})_1$. Indeed, any event $\underset{(3)}{x_r}$ of L_3 has the form $(r+h,0,0,ict_3)$. Then for $\underset{(3)}{x_r}-(\underset{(1)}{x_r})_1$ we obtain the value $(h,0,0,ict_3)$. If we now impose the condition that $(\underset{(1)}{\lambda_r})_1\cdot[\underset{(3)}{x_r}-(\underset{(1)}{x_r})_1]$ (which has the value $-c^2t_3$) must be null, we obtain for t_3 the value $t_3=0$; then, the event of L_3 which we have found is $(\underset{(3)}{x_r})_1=(r+h,0,0,0)$.

On the other hand, if we consider the event $(\underset{(1)}{x_r})_2$ of $L_1,(-r,0,0,ic\pi/\omega)$, we establish in a similar way that $(\underset{(1)}{\lambda_r})_2=(0,-r\omega,0,ic)$; then $\underset{(3)}{x_r}-(\underset{(1)}{x_r})_2$ has the value $(2r+h,0,0,ic(t_3-\pi/\omega))$ and the product $(\underset{(1)}{\lambda_r})_2\cdot[\underset{(3)}{x_r}-(\underset{(1)}{x_r})_2]$ has the value $-c^2(t_3-\pi/\omega)$, in such a way that the only value of t_3 that makes this product zero is π/ω.

Now, on the one hand we have $\left|\left(\underset{(3)}{x_r}\right)_1-\left(\underset{(1)}{x_r}\right)_1\right|^2=h^2$ and on the other $\left|\left(\underset{(3)}{x_r}\right)_2-\left(\underset{(1)}{x_r}\right)_2\right|^2=(2r+h)^2$. It is clear that these values are not equal because of (30).

As a consequence of this fact, the definition of rigid motion is not predetermined in the general case as happened in the previous cases with Definitions 7 and 9. In contrast we can decide between several options—the following two for instance.

Definition 11. The motion of a set of particles is rigid when each pair of particles are rigidly joined according to Definition 10.

Definition 12. The motion of a set of particles is rigid if there exist three particles rigidly joined in pairs according to Definition 10 and any one of the remaining particles is rigidly joined to these three according to the same definition.

The first definition does not give six degrees of freedom whereas the second, which does give six degrees of freedom to the motion, is not as natural an extension of the Newtonian definition as the first. Nevertheless, considerations such as the possibility of introducing the concept of rigid body or the number of degrees of freedom become irrelevant when trying to choose an option. This is because when one develops the concept of rigid motion, one deliberately leaves out (because it is possible and necessary) any relation with dynamics.

Simplicity can be a good reason for making a choice, and it is clear that Definition 11 is the simplest. For this reason it is interesting to generalize this definition to general relativity, and this can be done quite naturally. With a view to this, let us consider a four-dimensional differentiable manifold V_4, of class $\mathcal{C}^2$, with a symmetric tensor field $g(g_{ij})$ of class $\mathcal{C}^1$ in such a way that $\forall x \in V_4$, g_x is a nondegenerate indefinite bilinear form with signature $(3,1)$ on the tangent space T_x and with time orientation. Then we can give the following definition:

Definition 13. Consider two timelike world lines L and L'; let us say that L' is rigidly joined to L (we can write this as $L' \mathfrak{R} J_{(4)} L$) when for every arbitrary point $P'(x')$ of L', there exists a unique geodesic $\Gamma(\xi^i = \xi^i(t))$ passing through P' which intersects orthogonally with L in such a way that if P is the point of intersection between Γ and L, and if the parameter t is such that $\Gamma(t_0) \equiv P$, $\Gamma(t_1) \equiv P'$, then the integral

$$\Omega(P, P') = \tfrac{1}{2}(t_1 - t_2) \int_{t_0}^{t_1} g_{ij} u^i u^j \, dt \tag{31}$$

taken along Γ, with $u^i = d\xi^i/dt$, is constant.

In accordance with the previous observations, the following definition appears natural.

Definition 14. A timelike world tube τ in V_4 is rigid when for two arbitrary world lines L and L' of it, the integral (31) taken along the geodesic $\Gamma(\xi^i = \xi^i(t))$ passing through $P' \in L'$ which is orthogonal to L in P is independent of P and P'.

From this definition it is possible to derive the characterization given by Synge (1966). In fact if L is a given curve of a world tube τ and if $\lambda^i_{(a)}$ represents an orthonormal tetrad in $L(x^i = x^i(s))$ with $\lambda^i_{(4)} = A^i$ we know that

$$\frac{\delta\lambda^i_{(\alpha)}}{\delta s} = bA^i\lambda_{(\alpha)j}B^j \qquad (\alpha = 1,2,3) \tag{32}$$

where b represents the first curvature of L, and A^i, B^i are the unit tangent vector and the first normal vector, respectively. Then, denoting the origin of s by $P_0 \in L$, if P' is a point of τ in such a way that the geodesic orthogonal to L which passes through P' intersects L at P, and if s and σ represent the separations P_0P and PP' (along the geodesic), respectively, and μ^i is the unit tangent vector to the geodesic PP' at P, we will have

$$X^{(\alpha)} = \sigma\mu^i\lambda^{(\alpha)}_i \tag{33}$$

where $X^{(\alpha)}$ represents the Fermi coordinates of P' with respect to L. But $g^{ij}\Omega_i\Omega_j = X_{(\alpha)}X^{(\alpha)}$, where $\Omega_i = \partial\Omega/\partial x^i$, and then if τ is rigid we have $X_{(\alpha)}X^{(\alpha)} = \text{const}$, which is equivalent, as we know, to $\sigma_{ij} = 0$, where σ_{ij} is the rate of strain tensor.

On the other hand, using the shear tensor ρ_{ij}, we know that $\rho_{ij} = \sigma_{ij} - \frac{1}{3}\theta P_{ij}$ where θ is the expansion and P_{ij} the projection operator; then it is clear that $\sigma_{ij} = 0$ is equivalent to $\rho_{ij} = \theta = 0$, which is the definition given by Ehlers and Kundt (1962).

REFERENCES

Born, M. (1909). *Annalen der Physik*, **30** (4), 1.

Ehlers J., and Kundt, W. (1962). *Gravitation*, L. Witten, ed., p. 58. J. Wiley and Sons, New York.

Synge, J. L. (1966). *Relativity: The General Theory*, p. 173. North-Holland, Amsterdam.

Synge, J. L. (1972). *Relativity: The Special Theory*, p. 36. North-Holland, Amsterdam.